

Riemannian geodesics of semi Riemannian warped product metrics*

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Abstract

Let (M_1, g_1) and (M_2, g_2) be two C^∞ -differentiable connected, complete Riemannian manifolds, $k : M_1 \rightarrow \mathbb{R}$ a C^∞ -differentiable function, having $0 < k_0 < k(x) \leq K_0$, for any $x \in M_1$ and $g := g_1 - kg_2$ the semi Riemannian metric on the product manifold $M := M_1 \times M_2$.

We associate to g a suitable family of Riemannian metrics $G_r + g_2$, with $r > -K_0^{-1}$, on M and we call *Riemannian* geodesics of g the geodesics of g which are geodesics of a metric of the previous family, via a suitable reparametrization.

Among the properties of these geodesics, we quote:

For any $z_0 = (x_0, y_0) \in M$ and for any $y_1 \in M_2$ there exists a subset A of M_1 , such that all the geodesics of g joining z_0 with a point (x_1, y_1) , with $x_1 \in A$, are Riemannian. The Riemannian geodesics of g determine a "partial" property of geodesic connection on M . Finally, we determine two new classes of semi Riemannian metrics (one of which includes some FLRM-metrics), geodesically connected by Riemannian geodesics of g .

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1 Introduction

Let (M_1, g_1) and (M_2, g_2) be two connected, complete, Riemannian manifolds.

For the greater part of the paper, we shall use the assumption of the completeness of the two manifolds only to avoid to write a long and trivial series of inequalities.

Let $k : M_1 \rightarrow \mathbb{R}$ be a C^∞ -differentiable function, bounded from below away from zero.

We consider the semi Riemannian warped product metric $g : g_1 - kg_2$ and the family of Riemannian metrics $G_r + g_2$ on the manifold $M := M_1 \times M_2$, where $G_r := (k^{-1} + r)g_1$ and $r > -K_0^{-1} := k_1$, being $K_0 := \sup_{x \in M_1} \{k(x)\}$, if k is bounded from above and $r > 0 := k_1$ in the other case.

Then we prove that M is complete with respect to the metric $G_r + g_2$ and that the geodesics of $G_r + g_2$, belonging to a suitable subset, determine geodesics of g , via a suitable reparametrization, for any $r > k_1$.

We call them *Riemannian geodesics of g* .

We prove some properties of these geodesics and here we quote some of them as examples.

Let us consider $z_0 = (x_0, y_0) \in M$ and a geodesic $\zeta = (\gamma, \tau) : [0, 1] \rightarrow M$ of g , with $\gamma(0) = x_0$, $\tau(0) = y_0$, $\dot{\gamma}(0) = \tilde{X}$ and $\dot{\tau}(0) = \tilde{Y}$. If k is bounded and

$$g_1(\tilde{X}, \tilde{X}) > k(x_0)g_2(\tilde{Y}, \tilde{Y}) \frac{K_0 - k(x_0)}{K_0} ;$$

then ζ is a Riemannian geodesic of g .

An analogous statement holds, if k is unbounded from above.

A surprising property, being the Morse theory of Riemannian and semi Riemannian metrics quite different, is the following.

Since M_1 and M_2 are connected and complete with respect to the respective Riemannian metrics g_1 and g_2 , the manifold M_1 is *positive and negative geodesically connected with respect to g* ; i.e., for any real number $r > k_1$, for any $z_0 = (x_0, y_0) \in M$, for any $x_1 \in M_1$ and for any geodesic $\nu : \mathbb{R} \rightarrow M_2$ of g_2 , having $\nu(0) = y_0$, there exists $t_0 \in \mathbb{R}$ such that the point z_0 and the point $(x_1, \nu(t_0))$ (and the point $(x_1, \nu(-t_0))$) can be joined by a Riemannian geodesic of g , obtained by reparametrizing a suitable geodesic of $G_r + g_2$.

Analogously, the manifold M_2 is *positive and negative geodesically connected with respect to g* , too.

Hence, we shall say that M is *partially Riemannian connected with respect to g*

More surprising are the following two results.

If M_1 and M_2 are connected and complete with respect to the respective Riemannian metrics g_1 and g_2 , if the dimension of M_1 is greater than one and M_1 is simply connected, if g_1 has a negative sectional curvature, if k is bounded from below away from zero and if the Hessian of k verifies a suitable inequality (see (4.2), below), then M is geodesically connected by means Riemannian geodesic of g .

If $M_1 = \mathbb{R}$, then g is an FLRW-metric (with speed of light $c = 1$) and M is geodesically connected by Riemannian geodesic of g , provided M_2 connected and complete with respect to g_2 and k bounded from below away from zero.

The FLRW-metrics are used in cosmology to study the early universe (see, e. g., [9]).

The paper ends with an Appendix in which we determine a sufficient condition such that G_r has negative sectional curvature, for any $r \in (k_1, +\infty)$.

We conclude by noticing that the Levi-Civita connection of g is not used in this paper, because it hides all the relations between the metric tensor g and the Riemannian metric $G_r + g_2$.

In this case, the Levi-Civita connection of $g_1 + g_2$ allows us to use these relations.

Hence, we consider this paper as a first application of the results obtained in [1], [2] and [3].

2 Preliminaries

This Section contains the main geometric objects, which are needed in the following.

We also state some straightforward results.

Let (M_1, g_1) , (M_2, g_2) be two connected, complete, Riemannian manifolds and $\overset{1}{\nabla}$, $\overset{2}{\nabla}$ the Levi-Civita connections determined by the metrics g_1 and g_2 , respectively.

Let $k : M_1 \rightarrow \mathbb{R}$ be a smooth bounded map.

We suppose

$$0 < k_0 := \inf_{x \in M_1} \{k(x)\} . \quad (2.1)$$

On the manifold $M := M_1 \times M_2$, we consider the tensor $g := g_1 - k \cdot g_2$, which defines a *semi Riemannian warped product metric*, having the signature equal to the dimension of M_1 .

The geometry of warped product metrics is described in details in [7].

We shall set

$$G_r := \left(\frac{1}{k} + r\right) \cdot g_1$$

and G_r is a Riemannian metric on M_1 , for any $r > k_1$, being $k_1 := -K_0^{-1}$ if k is bounded and $K_0 := \sup_{x \in M_1} \{k(x)\}$, and $k_1 := 0$ in the other case.

Finally, we set $I := [0, 1]$.

From [3], it follows.

Lemma 2.1. *A differentiable curve $\zeta = (\gamma, \tau) : I \rightarrow M$ is a geodesic of g , if and only if it satisfies the following system of ordinary differential equations*

$$\overset{1}{\nabla}_{\dot{\gamma}} \dot{\gamma} = -\frac{1}{2} g_2(\dot{\tau}, \dot{\tau}) \cdot g_1^\sharp(dk) \circ \gamma \quad (2.2)$$

$$\overset{2}{\nabla}_{\dot{\tau}} \dot{\tau} = -\frac{1}{k \circ \gamma} dk(\dot{\gamma}) \cdot \dot{\tau} \quad (2.3)$$

where $g_1^\sharp : T^*M_1 \rightarrow TM_1$ is the canonical isomorphism of bundles induced by g_1 .

From [3], we also get:

Lemma 2.2. *The map $\mu : I \rightarrow M_1$ is a geodesic with respect to the metric G_r if and only if*

$$\overset{1}{\nabla}_{\dot{\mu}} \dot{\mu} = \frac{1}{2k \circ \mu(1 + rk \circ \mu)} \left\{ 2dk(\dot{\mu}) \cdot \dot{\mu} - g_1(\dot{\mu}, \dot{\mu}) \cdot g_1^\sharp(dk) \circ \mu \right\} . \quad (2.4)$$

We conclude this number by two lemmas needed in the following.

Lemma 2.3. *Let \mathcal{M} be a topological space equipped with two distance functions d_1 and d_2 . Suppose that any Cauchy sequence of d_2 is also a Cauchy sequence of d_1 . Then the completeness of d_1 implies the completeness of d_2 .*

A proof of the above lemma is straightforward and we omit it here.

We observe that if there exists a positive number L such that $d_1(x_1, x_2) \geq Ld_2(x_1, x_2)$, for each $x_1, x_2 \in \mathcal{M}$, then each Cauchy sequence of d_2 is also a Cauchy sequence of d_1 .

Corollary 2.1. *If the Inequality (2.1) holds, the manifold (M_1, g_1) is complete if and only if there exists an $r > k_1$ such that (M_1, G_r) is complete.*

Proof. We shall denote by d_{g_1} , d_{G_r} the distance functions associated with the Riemannian metrics g_1 and G_r , respectively.

For any $X \in T_{x_0}M_1$ and $x_0 \in M_1$, we have

$$g_1(X, X) = \frac{k(x_0)}{1 + rk(x_0)} G_r(X, X) \text{ and } G_r(X, X) = \frac{1 + rk(x_0)}{k(x_0)} g_1(X, X) ;$$

for any $r > k_1$.

The functions $f_1, f_2 : (k_0, +\infty) \rightarrow \mathbb{R}$ defined respectively by setting

$$f_1(t) = \frac{t}{1 + rt} \quad \text{and} \quad f_2(t) = \frac{1 + rt}{t} ; \quad \forall r \in (k_0, +\infty)$$

are bounded.

Hence, there exist two positive real numbers k_2 and k_3 such that

$$d_{g_1}(x_1, x_2) \leq \sqrt{k_2} d_{G_h}(x_1, x_2) \quad \text{and} \quad d_{G_r}(x_1, x_2) \leq \sqrt{k_3} d_{g_1}(x_1, x_2)$$

for all $x_1, x_2 \in M_1$.

Then our corollary follows immediately from Lemma 2.3.

□

Finally, we recall that connected, complete, Riemannian manifolds are geodesically connected (see, e. g., [5]).

3 Geodesics on $(M, G_r + g_2)$ and (M, g)

In this Section we shall use the geometric objects and the notations introduced in the previous one.

Lemma 3.1. *For any $\mu : I \rightarrow M_1$ and for any $r > k_1$, there is a uniquely determined diffeomorphism $\varphi_r : I \rightarrow I$ such that*

$$\begin{aligned} \varphi_r(0) &= 0, \quad \varphi_r(1) = 1 \\ \dot{\varphi}_r &= a_r \frac{1 + rk}{k} \circ \mu \circ \varphi_r \end{aligned} \tag{3.1}$$

where a_r is a suitable real number.

Proof. We shall determine φ_r^{-1} and then we shall obtain φ_r as the inverse of φ_r^{-1} .

Condition (3.1) is equivalent to

$$\frac{d\varphi_r^{-1}}{ds} = \frac{k(\mu(s))}{a_r(1 + rk(\mu(s)))}$$

Hence the map φ_r^{-1} is defined by

$$\varphi_r^{-1}(s) := \frac{1}{a_r} \int_0^s \frac{k}{1 + rk} \circ \mu \, d\xi \quad , \quad a_r := \int_0^1 \frac{k}{1 + rk} \circ \mu \, d\xi ; \quad (3.2)$$

for any $s \in I$.

As a consequence, φ_r^{-1} is a smooth strictly increasing diffeomorphism from I onto I .

□

We need the following lemma, too.

Lemma 3.2. *For any differentiable curve $\gamma : I \rightarrow M_1$, there is a uniquely determined diffeomorphism $\psi : I \rightarrow I$, such that*

$$\begin{aligned} \psi(0) &= 0, \quad \psi(1) = 1 \\ \dot{\psi} &= \frac{b}{k \circ \gamma} \, , \end{aligned} \quad (3.3)$$

where b is a suitable positive real number.

Proof. The map ψ is defined by

$$\psi(s) := b \int_0^s \frac{1}{k \circ \gamma} d\xi, \quad b := \left(\int_0^1 \frac{1}{k \circ \gamma} d\xi \right)^{-1} , \quad (3.4)$$

for any $s \in I$.

□

The previous lemma implies:

Theorem 3.1. *Let $\mu : I \rightarrow M_1$ and $\nu, \tau : I \rightarrow M_2$ be smooth curves and suppose $\tau = \nu \circ \psi$, being ψ defined by the previous lemma.*

Then, τ satisfies (2.3), if and only if ν is a geodesic of g_2 .

Moreover, it results $\tau(0) = \nu(0)$ and $\tau(1) = \nu(1)$.

Proof. In fact, it results

$$\begin{aligned}
\overset{2}{\nabla}_{\dot{\tau}} \dot{\tau} &= (\dot{\psi})^2 \cdot (\overset{2}{\nabla}_{\dot{\nu}} \dot{\nu}) \circ \psi + \ddot{\psi} \cdot \dot{\nu} \circ \psi = \\
&\stackrel{(3.3)}{=} (\dot{\psi})^2 \cdot (\overset{2}{\nabla}_{\dot{\nu}} \dot{\nu}) \circ \psi + \frac{b}{k^2 \circ \mu} ((dk)(\dot{\mu})) \cdot \dot{\nu} \circ \psi = \\
&= (\dot{\psi})^2 \cdot (\overset{2}{\nabla}_{\dot{\nu}} \dot{\nu}) \circ \psi - \frac{1}{k \circ \mu} dk(\dot{\mu}) \cdot \dot{\tau} ;
\end{aligned}$$

and we have the assertion. \square

Lemma 3.3. *Let $\mu_r, \gamma_r : I \rightarrow M_1$ be two smooth curves, such that $\gamma_r = \mu_r \circ \varphi_r$, being φ_r the mapping defined by Lemma 2.3, with $\mu = \mu_r$.*

Then, μ_r is a geodesic with respect to the metric G_r , if and only if the curve γ_r satisfies the equation:

$$\overset{1}{\nabla}_{\dot{\gamma}_r} \dot{\gamma}_r = \frac{-1}{2k \circ \gamma_r (1 + rk_r \circ \gamma_r)} g_1(\dot{\gamma}_r, \dot{\gamma}_r) g_1^\sharp(dk) \circ \gamma_r . \quad (3.5)$$

Moreover, we have $\mu_r(0) = \gamma_r(0)$ and $\mu_r(1) = \gamma_r(1)$.

Proof. In fact, we have

$$\begin{aligned}
\overset{1}{\nabla}_{\dot{\gamma}_r} \dot{\gamma}_r &= (\dot{\varphi}_r)^2 \cdot (\overset{1}{\nabla}_{\dot{\mu}_r} \dot{\mu}_r) \circ \varphi_r + \ddot{\varphi}_r \cdot (\dot{\mu}_r \circ \varphi_r) \\
&\stackrel{(2.4)}{=} \frac{-\dot{\varphi}_r^2}{2k \circ \mu_r \circ \varphi_r (1 + rk \circ \mu_r \circ \varphi_r)} g_1(\dot{\mu}_r, \dot{\mu}_r) \circ \varphi_r \cdot g_1^\sharp(dk) \circ \mu_r \circ \varphi_r \\
&\quad + \frac{\dot{\varphi}_r^2}{k_r \circ \mu_r \circ \varphi_r (1 + rk_r \circ \mu_r \circ \varphi_r)} dk(\dot{\mu}_r) \circ \varphi_r \cdot \dot{\mu}_r \circ \varphi_r \\
&\quad + \ddot{\varphi}_r \cdot \dot{\mu}_r \circ \varphi_r \\
&\stackrel{(3.2)}{=} \frac{-1}{2k_r \circ \gamma_r (1 + rk_r \circ \gamma_r)} g_1(\dot{\gamma}_r, \dot{\gamma}_r) g_1^\sharp(dk) \circ \gamma_r \\
&\quad + \frac{1}{k \circ \gamma_r (1 + rk \circ \gamma_r)} dk(\dot{\gamma}_r) \cdot \dot{\gamma}_r + \ddot{\varphi}_h(dk(\dot{\gamma}_h)) \cdot \dot{\mu}_h \circ \varphi_h \\
&\stackrel{(3.1)}{=} \frac{-1}{2k_r \circ \gamma_r (1 + rk_r \circ \gamma_r)} g_1(\dot{\gamma}_r, \dot{\gamma}_r) g_1^\sharp(dk) \circ \gamma_r .
\end{aligned}$$

Since the vice versa can be proved in an analogous way, our lemma follows. \square

Lemma 3.4. *Under the assumptions of the previous lemma, if either μ_r is a geodesic of G_r or γ_r verifies 3.5, we have*

$$g_1(\dot{\gamma}_r, \dot{\gamma}_r) = a_r^2 \frac{(1 + rk(x_0))(1 + rk \circ \gamma_r)}{k(x_0)k \circ \gamma_r} \cdot g_1(X_r, X_r) , \quad (3.6)$$

being $\gamma_r(0) = x_0$ and $X_r = \dot{\mu}_r(0)$.

Proof. In fact, it results

$$\begin{aligned} g_1(\dot{\gamma}_r, \dot{\gamma}_r) &= (\dot{\varphi}_r)^2 \cdot g_1(\dot{\mu}_r \circ \varphi_r, \dot{\mu}_r \circ \varphi_r) \\ &\stackrel{(3.2)}{=} a_r^2 \left(\frac{1 + rk}{k} \circ \mu_r \circ \varphi_r \right)^2 \cdot g_1(\dot{\mu}_r \circ \varphi_r, \dot{\mu}_r \circ \varphi_r) . \end{aligned}$$

Then, under the assumptions of our lemma, it follows

$$g_1(\dot{\gamma}_r, \dot{\gamma}_r) = a_r^2 \frac{1 + rk \circ \gamma_r}{k \circ \gamma_r} \cdot G_r(\dot{\mu}_r \circ \varphi_r, \dot{\mu}_r \circ \varphi_r) .$$

From which (3.6) immediately follows. □

From the above lemma and Lemma 3.3, we get the following

Lemma 3.5. *Under the assumptions of the previous lemma, if $\mu_r : I \rightarrow M_1$ is a geodesic with respect to the metric G_r then*

$$\overset{1}{\nabla}_{\dot{\gamma}_r} \dot{\gamma}_r = \frac{-a_r^2(1 + rk(x_0))}{2k(x_0)k^2 \circ \gamma_r} \cdot g_1(X_0, X_0) \cdot g_1^\sharp(dk) \circ \gamma_r . \quad (3.7)$$

The next lemma characterizes the norm of the vector field $\dot{\tau}_r$. We skip the proof of this lemma for it is very similar to that one of Lemma 3.4.

Lemma 3.6. *Let $\mu_r : I \rightarrow M_1$ and $\tau_r, \nu : I \rightarrow M_2$ be three smooth curves such that $\tau_r = \nu \circ \psi_r$, being ψ_r defined as in Lemma 3.2, by means of μ_r . If either ν_r is a geodesic of g_2 or τ_r is a solution of Equation 2.2, then*

$$g_2(\dot{\tau}, \dot{\tau}) = \frac{b_r^2}{k^2 \circ \gamma_r} \cdot g_2(Y_0, Y_0) ; \quad (3.8)$$

with $\nu(0) = y_0$ and $\dot{\nu}(0) = Y_0$.

With the previous notations, we have:

Theorem 3.2. *Suppose that the curve $(\mu_r, \nu_r) : I \rightarrow M$ is a geodesic with respect to the metric $G_r + g_2$ and*

$$a_r^2 \frac{1 + rk(x_0)}{k(x_0)} \cdot g_1(X_0, X_0) = b_r^2 g_2(Y_0, Y_0) ; \quad (3.9)$$

with $\mu_r(0) = x_0$, $\nu_r(0) = y_0$, $\dot{\mu}_r(0) = X_0$ and $\dot{\nu}_r(0) = Y_0$.

Then, the curve $(\gamma_r, \tau_r) : I \rightarrow M$, obtained as in the previous Lemmas is a geodesic with respect to the metric g .

We have $(\mu_r(0), \nu_r(0)) = (x_0, y_0)$ and $(\mu_r(1), \nu_r(1)) = (\gamma_r(1), \tau_r(1))$, too.

Proof. Since $(\mu_r, \nu_r) : I \rightarrow M$ is a geodesic of the metric $G_r + g_2$ then $\mu_r : I \rightarrow M_1$ is a geodesic of G_r and $\nu_r : I \rightarrow M_2$ is a geodesic of g_2 . Hence from Theorem 3.1 it follows that the curve (γ_r, τ_r) satisfies Equation (2.3).

As a consequence, we need only to prove that (γ_r, τ_r) satisfies Equation (2.2). In fact, we have

$$\begin{aligned} \overset{1}{\nabla}_{\dot{\gamma}_r} \dot{\gamma}_r &\stackrel{(3.7)}{=} \frac{-a_r^2(1 + rk(x_0))}{2k(x_0)k^2 \circ \gamma_r} \cdot g_1(X_0, X_0) \cdot g_1^\sharp(dk) \circ \gamma_r \\ &\stackrel{(3.9)}{=} \frac{-b_r^2}{2k^2 \circ \gamma_r} g_2(Y_0, Y_0) \cdot g_1^\sharp(dk) \circ \gamma_r \\ &\stackrel{(3.8)}{=} \frac{-1}{2} g_2(\dot{\tau}_r, \dot{\tau}_r) \cdot g_1^\sharp(dk) \circ \gamma_r . \end{aligned}$$

□

Hence, we put the following definition.

Definition 3.1. Let $(\mu_r, \nu_r) : I \rightarrow M$ be a geodesic of $G_r + g_2$ and let (γ_r, τ_r) be the geodesic of (M, g) obtained via the reparametrization by the functions φ_r and ψ_r from (μ_r, ν_r) .

Then, (γ_r, τ_r) is called *Riemannian geodesic of (M, g)* .

Remark 3.1. *Under the assumptions of the previous theorem we set:*

$$\mu_r(0) = x_0 = \gamma_r(0) , \quad \dot{\mu}_r(0) = X_0 = X_r , \quad \dot{\gamma}_r(0) = \tilde{X}_r \quad (3.10)$$

and

$$\nu_r(0) = y_0 = \tau_r(0) , \quad \dot{\nu}_r(0) = Y_0 = Y_r , \quad \dot{\tau}_r(0) = \tilde{Y}_r . \quad (3.11)$$

Then we have:

$$\tilde{X}_r = a_r \frac{1 + rk(x_0)}{k(x_0)} X_r \quad \text{and} \quad \tilde{Y}_r = \frac{b_r}{k(x_0)} Y_r . \quad (3.12)$$

With these notations, the first identity of 3.9 can be written as

$$a_r^2 \frac{1 + rk(x_0)}{k(x_0)} \cdot g_1(X_r, X_r) = b_r^2 g_2(Y_r, Y_r) ;$$

and it is equivalent to

$$g_1(\tilde{X}_r, \tilde{X}_r) = k(x_0)(1 + rk(x_0))g_2(\tilde{Y}_r, \tilde{Y}_r) . \quad (3.13)$$

The previous equality implies that the geodesic $(\hat{\nu}_r, \hat{\tau}_r)$ of g , having (x_0, y_0) and $(a\tilde{X}_r, a\tilde{Y}_r)$ as initial conditions, is a Riemannian geodesic of g , for any $a \in \mathbb{R}$.

From Equation (3.13) we get

Remark 3.2. Let $\zeta_r = (\gamma_r, \tau_r)$ and $\zeta_s = (\gamma_s, \mu_s)$ be two Riemannian geodesics of g , with $r, s > k_1$, such that $\zeta_r(0) = \zeta_s(0)$.

Then $\zeta_r = \zeta_s$, if and only if $r = s$.

Theorem 3.3. Suppose k bounded and let $\tilde{\zeta} = (\gamma, \tau) : I \rightarrow M$ be a geodesic of g , such that $\dot{\gamma}(0) = \tilde{X}_0$ and $\dot{\tau}(0) = \tilde{Y}_0 \neq 0$.

If

$$g_1(\tilde{X}, \tilde{X}) > k(x_0)g_2(\tilde{Y}, \tilde{Y}) \frac{K_0 - k(x_0)}{K_0} ; \quad (3.14)$$

the curve $\tilde{\zeta}$ is a Riemannian geodesic of g .

Proof. We set

$$r = \frac{g_1(\tilde{X}, \tilde{X})}{k^2(x_0)g_2(\tilde{Y}, \tilde{Y})} - \frac{1}{k(x_0)} .$$

Then a symple calculation shows that $r > k_1$.

Now we consider the curve τ and we set $\nu_r = \tau \circ \psi_r^{-1} : I \rightarrow M_2$, being ψ_r defined by γ as in Lemma 3.2.

Since the curve τ verifies Equation (2.3), the curve ν_r is a geodesic of g_2 .

Analogously, we set $\mu_r = \gamma \circ \varphi_r^{-1}$, with φ_r defined by Lemma 3.1, and μ_r is a geodesic of G_r , in the obvious way.

Finally, the previous construction implies that (γ, τ) is a Riemannian geodesic of g obtained from the geodesic (μ_r, ν_r) of $G_r + g_2$. □

Remark 3.3. *If k is unbounded from above and one replaces (3.14) by*

$$g_1(\tilde{X}, \tilde{X}) > k(x_0)g_2(\tilde{Y}, \tilde{Y}) ;$$

the previous theorem holds, again.

4 Some properties of Riemannian geodesics

Remark 4.1. *Let $\mu_r : I \rightarrow M_1$ be a geodesic of G_r , with $r > k_1$.*

We recall that there exist a geodesic $\sigma_r : \mathbb{R} \rightarrow M_1$ of G_r and $t_0 \in \mathbb{R}$ such that $(\sigma_r([0, t_0]) = \mu_r(I)$, being G_r a complete Riemannian metric.

Moreover, it results $\dot{\mu}_r(0) = t_0 \dot{\sigma}_r(0)$.

An analogous statement holds for g_2 .

This implies that the mappings φ_r and ψ_r defined respectively by Lemmas 3.1 and 3.2 can be extended to diffeomorphisms from \mathbb{R} onto \mathbb{R} .

Theorem 4.1. *Let $(\mu_r, \nu_r) : \mathbb{R} \rightarrow M$ be a geodesic of $G_r + g_2$, with $r > k_1$.*

Then, for any $\alpha \in \mathbb{R}$, there exist two real numbers $\pm\beta \in \mathbb{R}$, such that the point $(\mu_r(0), \nu_r(0))$ and the point $(\mu_r(\alpha), \nu_r(\pm\beta))$ can be joined by Riemannian geodesics of g .

Proof. We put $\dot{\mu}_r(0) = X_r$ and $\dot{\nu}_r(0) = Y_r$ and suppose $\|X_r\|_1 = \|Y_r\|_2 = 1$, with the obvious meaning of the used symbols and without loss of generality.

Then, for any $\alpha \in \mathbb{R}$ ($\beta \in \mathbb{R}$), the point $\mu_r(\alpha)$ ($\nu_r(\beta)$) is the end point of the geodesic of G_r (g_2), determined by the vector αX_r (βY_r).

We shall denote by $a_{\alpha r}$ and $b_{\alpha r}$ the constants of Lemmas 3.1 and 3.2 determined by means of the geodesic having $(x_0, \alpha X_r)$ as initial condition, respectively.

Then, $X_{\alpha r}$ and $Y_{\beta r}$ verify Condition (3.9), if and only if

$$a_{\alpha r}^2 \frac{1 + rk(x_0)}{k(x_0)} \alpha^2 = b_{\alpha r}^2 \beta^2 . \quad (4.1)$$

Hence, the assertion follows by computing β from (4.1). □

Theorem 4.2. *Let $(\mu_r, \nu_r) : \mathbb{R} \rightarrow M$ be a geodesic of $G_r + g_2$, with $r > k_1$.*

Then, for any $\beta \in \mathbb{R}$, there exist two real numbers $\pm\alpha \in \mathbb{R}$, such that the point $(\mu_r(0), \nu_r(0))$ and the point $(\mu_r(\pm\alpha), \nu_r(\beta))$ can be joined by Riemannian geodesics of g .

Proof. The proof is analogous to the previous one. □

Corollary 4.1. *For any $x_0, x_1 \in M_1$, for any $r > k_1$, for any geodesic $\mu_r : I \rightarrow M_2$ of G_r joining x_0 and x_1 and any geodesic $\nu_r : \mathbb{R} \rightarrow M_2$ of g_2 , there exists $\beta \in \mathbb{R}$ such that the points $(x_0, \nu(0))$ and $(x_1, \nu(\pm\beta))$ can be joined by a Riemannian geodesic of g , obtained in the obvious way from the previous two geodesics.*

An analogous statement holds for any $y_0, y_1 \in M_2$.

Definition 4.1. *Since Corollary 4.1 holds, we shall say that M_1 is positively and negatively geodesically connected with respect to g .*

Analogously, we shall say that M_2 is positively and negatively geodesically connected with respect to g .

Finally, we shall say that M is partially geodesically connected, when the previous two definitions hold.

Theorem 4.3. *Let us consider $x_0, x_1 \in M_1$ and let us suppose that there exists a continuous map $X : (k_1, +\infty) \rightarrow T_{x_0}M_1$, such that for any $r \in (k_1, +\infty)$ the geodesic $\mu_r : I \rightarrow M_1$ of G_r , determined by the initial condition $(x_0, X(r))$, joins x_0 and x_1 and that μ_r is minimizing.*

Then, for any $y_0, y_1 \in M_2$, there exists a Riemannian geodesic of g joining (x_0, y_0) and (x_1, y_1) .

Proof. Under the assumptions of the theorem, we consider the function $\beta : (k_1, \infty) \rightarrow \mathbb{R}$ defined by setting

$$\beta(r) = \frac{a_r}{b_r} \left(\frac{1 + rk(x_0)}{k(x_0)} \cdot g_1(X(r), X(r)) \right)^{\frac{1}{2}} ;$$

where a_r and b_r are obtained respectively by (3.2) and (3.4) along the geodesic $\mu_r : I \rightarrow M_1$ of G_r , for any $r \in (k_1, +\infty)$.

Then, β is continuous, too.

Let $\gamma : I \rightarrow M_1$ be a minimizing geodesic of g_1 joining x_0 and x_1 and let us set $\dot{\gamma}(0) = X$.

Since all the involved geodesics are minimizing, we have

$$g_1(X, X)a_r^{-1} \leq \frac{1 + rk(x_0)}{k(x_0)}g_1(X(r), X(r)) \leq g_1(X, X) \int_0^1 \frac{1 + rk(\gamma(t))}{k(\gamma(t))} dt$$

and

$$g_1(X, X)\frac{a_r}{b_r^2} \leq \beta^2(r) \leq g_1(X, X)\frac{a_r^2}{b_r^2} \int_0^1 \frac{1 + rk(\gamma(t))}{k(\gamma(t))} dt ;$$

The first of the previous inequalities and $k_0 > 0$ imply

$$\lim_{r \rightarrow k_1} \beta(r)^2 \geq \lim_{r \rightarrow -K_0^{-1}} \frac{k_0}{K_0^2(1 + rK_0)} = +\infty \quad \text{and} \quad \lim_{r \rightarrow +\infty} \beta(r)^2 = 0 .$$

Hence, it results

$$\lim_{r \rightarrow k_1} \beta(r) = +\infty \quad \text{and} \quad \lim_{r \rightarrow +\infty} \beta(r) = 0 .$$

As a consequence of the well known generalization of the Weistrass β is onto.

Now, we consider two points $y_0, y_1 \in M_2$.

If $y_0 = y_1$, the point (x_0, y_0) and the point (x_1, y_1) can be joined by a Riemannian geodesic of g in a trivial way.

Suppose $y_0 \neq y_1$, then there exists a geodesic $\nu : \mathbb{R} \rightarrow M_2$ of g_2 and there exists $\beta_0 \in (0, +\infty)$, such that $\nu(0) = y_0$, $g_2(\dot{\nu}(0), \dot{\nu}(0)) = 1$ and $\nu(\beta_0) = y_1$.

Then, the geodesic of g_2 having $(y_0, \beta_0 \dot{\nu}(0))$ joins y_0 and y_1 .

Finally, we can consider $r_0 \in (k_1, +\infty)$ such that $\beta(r_0) = \beta_0$. With this choice the vectors X_{r_0} and $Y_{r_0} = \beta(r_0)\dot{\nu}(0)$ verify (3.9) and the assertion follows in a trivial way.

□

Theorem 4.4. *Suppose that the manifold M_1 is connected, has dimension higher than one, negative sectional curvature and that it is simply connected.*

Suppose that M_2 is connected, too.

Moreover, suppose that $k_0 > 0$ and that

$$(\overset{1}{\nabla} dk)(e, e) < \frac{1 + 4rk}{2k(1 + rk)}e(k)^2 + \frac{1}{4k(1 + rk)}\|dk\|_1^2 - k(1 + rk)\overset{1}{K}(\sigma) ; (4.2)$$

for any vector $e \in T_x M_1$, such that $g_1(e, e) = 1$ and for any $x \in M_1$.

If M_1 and M_2 are geodesically connected with respect to the metrics g_1 and g_2 , respectively, then for any $z_0, z_1 \in M$ there exists a Riemannian geodesic of g joining z_1 and z_2 .

Proof. From the Appendix it follows that the sectional curvature of G_r is negative, for any $r > k_1$.

Since M_1 is simply connected, the exponential mapping of G_r , $\exp_x^r : T_x M_1 \rightarrow M_1$, is a diffeomorphism, for any $x \in M_1$ (see, e. g. [5]).

Because of a theorem on the families of systems of ordinary differential equations continuously depending on a parameter, \exp_x^r is continuous with respect to $r > k_1$, too.

Let us consider $x_0, x_1 \in M_1$ and the map $X : (k_1, \infty) \rightarrow T_{x_0} M_1$ defined by setting $X(r) = (\exp_{x_0}^r)^{-1}(x_1)$, for any $r \in (k_1, \infty)$.

Then, X is continuous and the assertion follows from the previous theorem. □

Remark 4.2. Obviously, under the assumption of the previous theorem, for r tending to k_1 the contribution of $k(\sigma)$ is zero, but the contribution of the second summand tends to $+\infty$.

Suppose that $M_1 = \mathbb{R}$ and that $g_1 = dt^2$ is the standard metric on \mathbb{R} .

In this case the metric $g = dt^2 - k(t)g_2$ coincides with the FLRW-metric (Friedman–Lemaitre–Robertson–Walker metric), with speed of light $c = 1$, used in the Big Bang theories and we have:

Theorem 4.5. *If M_2 is complete with respect to the metric g_2 and k is bounded from above and bounded from below away from zero, then for any $z_0, z_1 \in M = \mathbb{R} \times M_2$ there exists a Riemannian geodesic of g joining z_1 and z_2 .*

Proof. In this case, the metric tensor G_r on \mathbb{R} is given by $G_r = (k^{-1} + r)dt^2$, for any $r > -K_0^{-1}$.

Let be $r > -K_0^{-1}$, then the Equation (2.4) of a geodesic of G_r becomes

$$\ddot{\mu}_r = \frac{1}{2(k \circ \mu_r)(1 + rk \circ \mu_r)}(k' \circ \mu_r)\dot{\mu}_r^2$$

The previous equation admits a first integral given by

$$\dot{\mu}_r = c_r \left(\frac{k \circ \mu_r}{1 + rk \circ \mu_r} \right)^{\frac{1}{2}}.$$

Because of Corollary 2.1, \mathbb{R} is complete with respect to the metric G_r .

Hence, we can determine c_r as a solution of the equation

$$c_r = (x_1 - x_0) \left(\int_0^1 \left(\frac{k(\mu_r(t))}{1 + rk(\mu_r(t))} \right)^{\frac{1}{2}} dt \right)^{-1}.$$

As a consequence, the mapping $\mu_r : I \rightarrow \mathbb{R}$ is strictly increasing, for $x_1 > x_0$ and strictly decreasing, for $x_1 < x_0$, because the function k is bounded from below by $k_0 > 0$.

This implies that $\exp_{x_0}^r : \mathbb{R} \rightarrow \mathbb{R}$ is a diffeomorphism.

Since $\exp_{x_0}^r$ depends with continuity from $r \in (-K_0^{-1}, +\infty)$, the proof follows as in the previous case. \square

Remark 4.3. *The previous theorem holds again, if one replaces the metric dt^2 on \mathbb{R} by the Riemannian metric $f dt^2$, being $f : \mathbb{R} \rightarrow \mathbb{R}$ a C^∞ -differentiable function such that $0 < f(t) < c$, for any $t \in \mathbb{R}$, with $c \in \mathbb{R}$.*

Now we return to the general case.

Theorem 4.6. *Let us consider $r \in (k_1, +\infty)$, a geodesic $\mu_r : \mathbb{R} \rightarrow M_1$ of G_r and a geodesic $\nu_r : \mathbb{R} \rightarrow M_2$ of g_2 .*

If $(\mu_r)|_{[0, +\infty)}$ has no auto intersections, there exists a map $\theta_r : \mu_r([0, +\infty)) \rightarrow \nu_r([0, +\infty))$, such that the points $(\mu_r(0), \nu_r(0))$ and $(\mu_r(t), \theta_r(\nu_r(t)))$, can be joined by a Riemannian geodesic of g obtained from the geodesic $(\mu_r, \nu_r) : \mathbb{R} \rightarrow M$ of $G_r + g_2$ in the obvious way and the mapping θ_r is onto.

Moreover, if ν_r has no auto intersections, the mapping θ_r is one to one, too.

Proof. Under the assumptions of the theorem, we set $\dot{\mu}_r(0) = X_r$, $\dot{\nu}_r(0) = Y_r$ and we suppose $\|X_r\|_1 = \|Y_r\|_2 = 1$.

We recall that, for any $t \in \mathbb{R}$, the geodesic μ'_r of G_r determined by the initial conditions $(\mu_r(0), tX_r)$, has $\mu'_r(I) \subseteq \mu_r(\mathbb{R})$, joins $\mu_r(0)$ and $\mu_r(t)$ and the obvious quantities a'_r and b'_r are

$$a'_r := \int_0^1 \frac{k}{1 + rk} \circ \mu(\xi t) d\xi \quad \text{and} \quad (b'_r)^{-1} = \int_0^1 \frac{1}{k \circ \mu_r(\xi t)} d\xi.$$

An analogous statement holds for ν_r .

Now, we notice that, since μ_r has no autointersections, we can consider the map $\mu_r^{-1} : \mu_r([0, +\infty)) \rightarrow [0, +\infty)$.

Moreover, the Condition (3.9) determines the mapping $\beta : [0, +\infty) \rightarrow \mathbb{R}$, defined by:

$$\beta(t) = \frac{a'_r}{b'_r} \frac{1 + rk(x_0)}{k(x_0)} t, \quad \forall t \in [0, +\infty) .$$

□

Then, we can set $\theta_r = \nu_r \circ \beta \circ \mu_r^{-1} : \mu_r([0, +\infty)) \rightarrow \nu_r([0, +\infty))$.

Let us consider $x_1 \in \mu_r([0, +\infty))$, then exists $t \in [0, +\infty)$, such that $\mu_r(t) = x_1$, hence $t = \mu_r^{-1}(x_1)$.

Then, $\beta(t)$ is such that the vectors tX_r and $\beta(t)Y_r$ verify (3.9).

As a consequence, the points $(\mu_r(0), \nu_r(0))$ and $(\mu_r(t), \nu_r(\beta(t)))$ can be joined by a Riemannian geodesic for g obtained from the geodesic (μ_r, ν_r) of $G_r + g_2$, with $\nu_r(\beta(t)) = \nu_r(\beta(\mu_r^{-1}(x_1))) = \theta_r(x_1)$.

5 Appendix

In this Appendix we prove the following lemma:

Lemma 5.1. *Suppose that the dimension of M_1 is higher than one and that g_1 has negative sectional curvature.*

Then, if k verifies (4.2), G_r has negative sectional curvature, for any $r \in (k_1, +\infty)$.

Proof. Let $\Xi(M_1)$ be the Lie algebra of vector fields on M_1 .

Let us consider a connection ∇^h of M_1 and let us suppose $\nabla^h = \overset{1}{\nabla} + \Pi$.

Then, the curvature tensor field R^h of ∇^h and the curvature tensor field $\overset{1}{R}$ of $\overset{1}{\nabla}$ are related by

$$\begin{aligned} R^h(X, Y)Z &= \overset{1}{R}(X, Y)Z \\ &+ \Pi(X, \Pi(Y, Z)) - \Pi(Y, \Pi(X, Z)) + \\ &(\overset{1}{\nabla}_X \Pi)(Y, Z) - (\overset{1}{\nabla}_Y \Pi)(X, Z), \quad \forall X, Y, Z \in \Xi(M_1) . \end{aligned}$$

Now we suppose that $h : M_1 \rightarrow \mathbb{R}$ is a C^∞ -differentiable function and that $h(x) > 0$, for any $x \in M_1$.

We also suppose that ∇^h is the Levi-Civita connection of the metric tensor hg_1 .

Then, we have

$$\begin{aligned} \Pi(X, Y) &= \frac{1}{2h} \left[X(h)Y + Y(h)X - g_1(X, Y)g_1^\sharp(d \log h) \right] \\ &\quad \forall X, Y \in \Xi(M_1) . \end{aligned}$$

The two previous identities imply

$$\begin{aligned} R^h(X, Y)Z &= \overset{1}{R}(X, Y)Z + \\ &\frac{1}{2h} [(\overset{1}{\nabla} dh)(X, Z)Y - (\overset{1}{\nabla} dh)(Y, Z)X - \\ &g_1(Y, Z)g_1^\sharp(\overset{1}{\nabla}_X dh) + g_1(X, Z)g_1^\sharp(\overset{1}{\nabla}_Y dh)] - \\ &\frac{1}{4h^2} [3Y(h)Z(h)X - 3X(h)Z(h)Y - \\ &Y(h)g_1(X, Z)g_1^\sharp(dh) + X(h)g_1(Y, Z)g_1^\sharp(dh) + \\ &g_1(Y, Z)\|d \log h\|_1^2 X - g_1(X, Z)\|d \log h\|_1^2 Y] , \quad \forall X, Y, Z \in \Xi(M_1) . \end{aligned}$$

Let $\sigma = \langle \{e_1, e_2\} \rangle$ be a two dimensional subspace of $T_x M_1$, with $x \in M_1$ and let us suppose $\|e_1\|_1 = \|e_2\|_1 = 1$ and $g_1(e_1, e_2) = 0$.

Then, the sectional curvature of ∇^h is

$$\begin{aligned} K(\sigma) &= \frac{1}{h} \overset{1}{K}(\sigma) - \frac{1}{2h^2} [(\overset{1}{\nabla} dh)(e_1, e_1) + (\overset{1}{\nabla} dh)(e_2, e_2)] - \\ &\frac{1}{4h^3} [3e_1(h)^2 + 3e_2(h)^2 - \|dh\|_1^2] ; \end{aligned}$$

being $\overset{1}{K}$ the sectional curvature of $\overset{1}{\nabla}$.

Now we suppose $h = k^{-1} + r$, where k is the mapping used in the previous numbers and $r > K_0^{-1} = k_1$.

Then, $dh = -k^{-2}dk$ and $\overset{1}{\nabla} dh = 2k^{-3}dk \otimes dk - k^{-2}\overset{1}{\nabla} dk$.

Hence, the sectional curvature of G_r is

$$\begin{aligned}
K_r(\sigma) = & \frac{k}{1+rk} \overset{1}{K}(\sigma) + \frac{1}{2(1+rk)^2} [(\overset{1}{\nabla} dk)(e_1, e_1) + (\overset{1}{\nabla} dk)(e_2, e_2)] - \\
& \frac{1+4rk}{4k(1+rk)^3} [e_1(k)^2 + e_2(k)^2] - \\
& \frac{1}{4k(1+rk)^3} \|dk\|_1^2 ;
\end{aligned}$$

for any two dimensional subspace $\sigma \subseteq T_x M_1$, for any (e_1, e_2) basis of σ such that $\|e_1\|_1 = \|e_2\|_1 = 1$ and $g_1(e_1, e_2) = 0$ and for any $x \in M_1$.

As a consequence, the sectional curvature of G_r is negative, for any $r > k_1$, if and only if

$$\begin{aligned}
& (\overset{1}{\nabla} dk)(e_1, e_1) + (\overset{1}{\nabla} dk)(e_2, e_2) < \\
& \frac{1+4rk}{2k(1+rk)} [e_1(k)^2 + e_2(k)^2] + \\
& \frac{1}{2k(1+rk)} \|dk\|_1^2 - 2k(1+rk) \overset{1}{K}(\sigma) ;
\end{aligned} \tag{5.1}$$

We notice that, if the sectional curvature $\overset{1}{K}$ of g_1 is positive, then the Inequality (5.1) can not hold for any $r > k_1$.

Hence, we are forced to suppose the g_1 has either a negative or null sectional curvature.

In this case, the Inequality (5.1) holds, if and only if, it results

$$(\overset{1}{\nabla} dk)(e, e) < \frac{1+4rk}{2k(1+rk)} e(k)^2 + \frac{1}{4k(1+rk)} \|dk\|_1^2 - k(1+rk) \overset{1}{K}(\sigma) ; \tag{5.2}$$

for any $e \in T_x M_1$, such that $g_1(e, e) = 1$ and any $x \in M_1$.

□

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